Homework 4 Solutions

Based on the third edition of *Abstract Algebra* by Dummit and Foote.

3.1 Definitions and Examples

Section 3.1, Exercise 1

Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \le G$ (i.e., the preimage or pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker\varphi \trianglelefteq G$.

Solution. Because $1_H \in E$ and $\varphi(1_G) = 1_H$, the set $\varphi^{-1}(E)$ is nonempty. If $a, b \in \varphi^{-1}(E)$, then $\varphi(a), \varphi(b) \in E$, and since E is a subgroup,

$$\varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1} \in E.$$

Thus $ab^{-1} \in \varphi^{-1}(E)$. The subgroup criterion gives $\varphi^{-1}(E) \le G$.

Now suppose $E \trianglelefteq H$. If $g \in G$ and $x \in \varphi^{-1}(E)$, then

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} \in E,$$

because $\varphi(x) \in E$ and E is normal in H. Hence $gxg^{-1} \in \varphi^{-1}(E)$ for all such g, x, so $\varphi^{-1}(E) \trianglelefteq G$.

Finally, $\{1_H\} \trianglelefteq H$ and

$$\ker\varphi = \varphi^{-1}(\{1_H\}).$$

Applying the preceding conclusion with $E = \{1_H\}$ yields $\ker\varphi \trianglelefteq G$. □

Section 3.1, Exercise 22

(a) Prove that if H and K are normal subgroups of a group G then their intersection $H \cap K$ is also a normal subgroup of G.

(b) Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

Solution.

(a) Since an intersection of subgroups is a subgroup, $H \cap K \le G$. For every $g \in G$ and $x \in H \cap K$, $gxg^{-1} \in H, K$ whence $gxg^{-1} \in\in H \cap K$. Therefore $H \cap K \trianglelefteq G$.

(b) Let $\{N_i\}_{i\in I}$ be a nonempty collection of normal subgroups of G, where the index set I is arbitrary, and put

$$N = \bigcap_{i\in I} N_i.$$

The identity belongs to every N_i, so N is nonempty. If $a, b \in N$, then $a, b \in N_i$ for every i, hence $ab^{-1} \in N_i$ for every i. Thus $ab^{-1} \in N$, proving $N \le G$.

For $g \in G$ and $x \in N$, one has $x \in N_i$ for every i. Normality of each N_i gives $gxg^{-1} \in N_i$ for every i, and consequently $gxg^{-1} \in N$. Hence $N \trianglelefteq G$.

□

Section 3.1, Exercise 24

Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

Solution. The intersection $N \cap H$ is a subgroup of H. Let $h \in H$ and $x \in N \cap H$. Since $N \trianglelefteq G$ and $h \in G$, we have $hxh^{-1} \in N$. Since $x, h \in H$ and H is a subgroup, we also have $hxh^{-1} \in H$. Therefore $hxh^{-1} \in N \cap H$ whence $N \cap H \trianglelefteq H$. □

Section 3.1, Exercise 36

Prove that if $G/Z(G)$ is cyclic then G is abelian. [If $G/Z(G)$ is cyclic with generator $xZ(G)$, show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

Solution. Assume $G/Z(G) = \langle xZ(G) \rangle$. Let $g \in G$. Since the coset $gZ(G)$ is a power of the generator, there is an integer a such that

$$gZ(G) = (xZ(G))^a = x^a Z(G).$$

Thus $g = x^a z$ for some $z \in Z(G)$, as suggested in the hint.

Now take arbitrary $g, h \in G$. Write $g = x^a z$ and $h = x^b w$ with $a, b \in \mathbb{Z}$ and $z, w \in Z(G)$. Since z and w commute with every element of G,

$$gh = x^a z x^b w = x^{a+b} zw = x^{b+a} wz = x^b w x^a z = hg.$$

Every pair of elements of G commutes; therefore G is abelian. □

Section 3.1, Exercise 38

Let A be an abelian group and let D be the (diagonal) subgroup $\{(a, a) \mid a \in A\}$ of $A \times A$. Prove that D is a normal subgroup of $A \times A$ and $(A \times A)/D \cong A$.

Note: The assertion is false without the "abelian" assumption, see January 2014 Qualifier, Problem 2.

Solution. First, D is a subgroup: $(1, 1) \in D$, and

$$(a, a)(b, b)^{-1} = (ab^{-1}, ab^{-1}) \in D.$$

Because A is abelian, the direct product $A \times A$ is abelian. Every subgroup of an abelian group is normal, so $D \trianglelefteq A \times A$.

Define

$$\psi : A \times A \longrightarrow A, \qquad \psi(a, b) = ab^{-1}.$$

For $(a, b), (c, d) \in A \times A$, commutativity in A gives

$$\psi((a, b)(c, d)) = \psi(ac, bd) = ac(bd)^{-1} = ab^{-1}cd^{-1} = \psi(a, b)\psi(c, d),$$

so ψ is a homomorphism. It is surjective because $\psi(a, 1) = a$ for every $a \in A$. Moreover,

$$\ker \psi = \{(a, b) \in A \times A : ab^{-1} = 1\} = \{(a, a) : a \in A\} = D.$$

The First Isomorphism Theorem therefore gives $(A \times A)/D \cong A$. □

Section 3.1, Exercise 40

Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the commutator of x and y and is denoted by $[x, y]$.)

Solution. Let $\overline{x} = xN$ and $\overline{y} = yN$. We have

$$\overline{x}\,\overline{y} = \overline{y}\,\overline{x} \iff \overline{x}^{-1}\overline{y}^{-1}\overline{x}\,\overline{y} = \overline{1} \iff x^{-1}y^{-1}xy \in N,$$

which finishes the proof. □

Section 3.1, Exercise 41

Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x, y \in G\rangle$ is a normal subgroup of G and G/N is abelian (N is called the commutator subgroup of G).

Solution. Let

$$S = \{x^{-1}y^{-1}xy : x, y \in G\}, \qquad N = \langle S \rangle\,.$$

By definition, N is a subgroup of G. For $g \in G$ and a generator $[x, y] = x^{-1}y^{-1}xy \in S$, we have

$$\begin{aligned} g[x,y]g^{-1} &= gx^{-1}y^{-1}xyg^{-1} \\ &= (gxg^{-1})^{-1}(gyg^{-1})^{-1}(gxg^{-1})(gyg^{-1}) \\ &= [gxg^{-1}, gyg^{-1}] \in S. \end{aligned}$$

Therefore $N \trianglelefteq G$.

Finally, every commutator $x^{-1}y^{-1}xy$ belongs to N by construction. Exercise 40 now shows that the cosets xN and yN commute for every $x, y \in G$. Hence every two elements of G/N commute, and G/N is abelian. □